SOME SELF-SIMILAR PROBLEMS OF STAR CRACK DEVELOPMENT UNDER ANTIPLANE DEFORMATION CONDITIONS

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UDC 539.30

INTRODUCTION

A self-similar plane problem of the theory of elasticity about a system of radial cracks, distributed uniformly in angle, being developed from a point at a constant velocity under antiplane deformation conditions is solved by the Smirnov–Sobolev method of functionally invariant solutions [1-4].

The formulation under investigation can be considered as a model for the mathematically more complex problem of plane deformation.

In an unloaded x, y plane let a system of 2n radial slits start to be developed from the origin at a constant speed at the initial instant. The slit edges are loaded in such a way that the whole elastic space is subjected to antiplane deformation along the z axis. Hence, only $w=w(x, y, t)$ – the z axis component of the displacement vector – is not zero. In this case the nonzero components of the stress tensor have the following form:

$$\tau_{yz} = \mu\partial w/\partial y;\ \tau_{xz} = \mu\partial w/\partial x.$$

The function w satisfies the wave equation

$$\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 = b^{-2}\partial^2 w/\partial t^2. \tag{0.1}$$

Let us examine a region of the x, y plane bounded by rays passing through adjacent slits and the arc of a shear wave (Fig. 1). The solution of (0.1) which will satisfy certain boundary conditions is sought in this domain. On the edges of the slits $\rho=\sqrt{x^2+y^2}<vt$; $\varphi=\arctan(y/x)=0; \pi/n$ stresses are given, while compliance with the condition $w=0$ is required on the sections $vt<\rho<bt$; $\varphi=0$. The latter results in some symmetry conditions for the effective loads depending on the evenness or oddness of w relative to the angle bisector. The boundary conditions on the wave depend on the kind of load and will henceforth be mentioned separately in each problem.

§1. Let us examine the case when the load on the slit edges is given in the form $\mathbf{p}=p_0 f(\rho/bt)\cdot\mathbf{k}$ (k is the unit vector of the z axis). This kind of loading corresponds to the Broberg problem [5]. As is known [2], the stress-tensor components τ_{xy}, τ_{yz} and the rate of displacement $\dot{w}$ are hence homogeneous functions of the coordinates and time of zero degree. Using the method of functionally invariant solutions of the wave equation, we find $\dot{w}$ in the form

$$\dot{w} = \operatorname{Re} U(z_2),\ z_2 = [\operatorname{ch}(n(\beta - i\varphi))]^{-1};\ \operatorname{ch}\beta = bt/\rho,$$

where $U(z_2)$ is some analytic function.

The domain shown in Fig. 1 goes over into the upper half-plane $y_2 \geq 0$ in the $z_2=x_2+iy_2$ plane. The edges of the slits along the rays go over into segments $(0, x_{21})$ and $(-x_{21}, 0)$ of the x_2 axis, respectively, for $\varphi=0$ and $\varphi=\pi/n$, where $x_{21}=[\cosh(n\operatorname{arccosh}(b/v))]^{-1}$. The arc of the wave $\rho=bt$, $0\leq\varphi\leq\pi/n$ goes over into the rays $(1, +\infty)$, $(-\infty, -1)$. At infinity of the z_2 plane is the point of intersection between the bisector of the angle π/n and the arc of the wave.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 168-174, September-October, 1976. Original article submitted July 17, 1975.

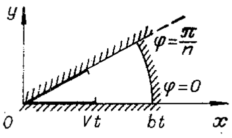

Fig. 1

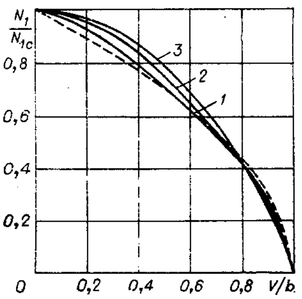

Fig. 2

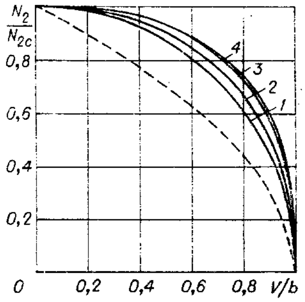

Fig. 3

Let us find the boundary conditions which the function $U(z_2)$ satisfies at $y_2=0$. Since the body is assumed to be in the rest state ahead of the wave, $\dot{w}=0$ on the wave [1] or $\dot{w}=\mathrm{Re}\ U(z_2)=0$ at $y_2=0$, $-\infty<x_2<-1$, $+1<x_2<+\infty$. The stresses $\varphi=0$ and $\varphi=\pi/n$, $w=0$ at $w=\mathrm{Re}\ U(z_2)=0$ at $y_2=0$, $x_{21}<|x_2|<1$, act at the edges of the slits, from which

$$\mathrm{Im}\ U'(z_2)=(4p_0/\mu)br(1+r^2)^{-2}f'(\xi)|dz_1/dz_2| \quad \text{for } y_2=0,\ \ 0<|x_2|<x_{21};$$

here

$$z_2=2z_1^n\left(1+z_1^{2n}\right)^{-1};\ z_1=re^{i\varphi};\ r=\xi^{-1}\left[1-\sqrt{1-\xi^2}\right];\ \xi=\rho/bt.$$

Therefore, we obtain a mixed boundary-value problem for the function $U'(z_2)$. Here $\mathrm{Im}\ U'(z_2)$ is given in the interval $(-x_{21}, +x_{21})$ and $\mathrm{Re}\ U'(z_2)=0$ outside. The general solution of this problem can be written down by using the Keldysh–Sedov formula [6]. Let us examine in detail a particular form of the loading $f(\xi)=\mathrm{const}$. Here, as in the general case, two versions are possible, namely: $\dot{w}$ is an even function of x_2, which corresponds to "folding of the angle," or $\dot{w}$ is an odd function of x_2, which corresponds to "torsion of the angle." Let us require $\dot{w}$ to be bounded at the vertex of the angle and at the intersection between the bisector of the angle π/n and the arc of the wave.

In the first case, we find analogously to [2] that the function

$$U'(z_2)=Aiz_2\left(z_2^2-x_{21}^2\right)^{-3/2}$$

satisfies the boundary conditions, has the required order of the singularity at the noses of the slits, and has the correct behavior at the points $z_2=0$ and $z_2=\infty$, corresponding to the vertex of the angle 0 under investigation and the point of intersection of the bisector of the angle π/n with the arc of the wave in the physical plane (see Fig. 1).

The coefficient A is found from the condition that $\tau_{yz}=p_0$ on the edge of the slit $0<x_2<x_{21}$. In this problem $\tau_{yz}(x, y, t)$, exactly as $\dot{w}(x, y, t)$, is a homogeneous function of zero measurement which satisfies the wave equation. Hence, by using the method of functionally invariant solutions it can be represented as

$$\tau_{yz}=\mathrm{Re}\ T(z_2), \tag{1.1}$$

where $T(z_2)$ is some analytic function.

The equality $\dot{\tau}_{yz}=\mu\,\partial\dot{w}/\partial y$ permits relating the functions $T(z_2)$ and $U(z_2)$:

$$(dT/dz_2)(dz_2/dz)(dz/dt)=\mu(dU/dz_2)(dz_2/dz)(dz/dy),$$

where $z=b^{-1}\cosh(n^{-1}\operatorname{arccosh} z_2^{-1})$. Hence, for dT/dz_2 we obtain

$$dT/dz_2=-Aiz_2\mu\sqrt{b^{-2}-z^2}\cdot\left(z_2^2-x_{21}^2\right)^{-3/2}, \tag{1.2}$$

where by integrating along the contour passing along the upper edge of the slit $(x_{21}, 1)$ and bypassing the point $z_2=x_{21}$ around an infinitesimal semicircle, we obtain

$$p_0=A\mu/v\int\limits_{v/b}^{1}\frac{z\,dz}{\sqrt{z^2-(v/b)^2}\sqrt{[\mathrm{ch}\ (n\,\mathrm{arch}\,(bz/v))]^{-2}-x_{21}^2}}=A\mathcal{J}_1\mu/v. \tag{1.3}$$

Using the equalities (1.1)–(1.3), we find that τ_{yz} near the nose of the slit $\tau_{yz}\simeq N_1(x-vt)^{-1/2}$ has the asymptotic for $\varphi=0$, $(x-vt)/(vt)\ll 1$, where

$$N_1/N_{1c} = \sqrt{\pi}/2 \cdot \mathcal{J}_1^{-1} \frac{\Gamma(1/2+1/2n)}{\Gamma(1+1/2n)} \frac{\sqrt{1-(v/b)^2}}{(1+a^2)a^{n-1/2}} \sqrt{\frac{b}{2v}} n^{-1} \sqrt{\frac{(1+a^{2n})^3(1-a^2)}{1-a^{2n}}}, \tag{1.4}$$

$$a = b/v \cdot [1 - \sqrt{1-(v/b)^2}].$$

Hence, $N_{1c} = p_0\sqrt{2n/\pi} \cdot \Gamma(1+1/2n)\Gamma^{-1}(1/2+1/2n)\sqrt{l}$ is the value of the stress-intensity coefficient for a singularity at the nose of the slit in the static problem about loading a star crack with slits of length $l = vt$ for an analogous loading by the shear stress p_0. The expression (1.4) is simplified substantially for $n=1$, which corresponds to an isolated crack:

$$N_1/N_{1c} = E^{-1}(\sqrt{1-(v/b)^2})\sqrt{1-(v/b)^2};\ N_{1c} = p_0\sqrt{l/2},$$

where $E(x)$ is the complete elliptic integral of the second kind.

In case of loading the sides of the angle by a load of different sign, the displacement rate $\dot{w}$ is an odd function with respect to the angle bisector or the $x_2=0$ axis in the z_2 plane. Then the function dU/dz_2, which satisfies all the boundary conditions and the additional conditions at the points $z_2=0$ and $z_2=\infty$, will be

$$dU/dz_2 = A_1 i\left(z_2^2 - x_{21}^2\right)^{-3/2},\quad A_1 = p_0 x_{21}^2 b(\mu\mathcal{J}_2)^{-1},$$

$$\mathcal{J}_2 = \int_1^{b/v} \frac{z\,dz}{\sqrt{z^2-1}\sqrt{1-x_{21}^2\operatorname{ch}^2(n\operatorname{arch} z)}}.$$

Here the coefficient A_1 is found from the same conditions as above. For a singularity in the stress field at the noses of the slits, the coefficient referred to the corresponding static value equals

$$N_2/N_{2c} = \sqrt{\pi}/\mathcal{J}_2 \cdot \frac{\Gamma(1+1/2n)}{\Gamma(1/2+1/2n)} \sqrt{x_{21}a^{1-n}}(v/b)^{-3/2} \frac{1+a^{2n}}{1+a^2} \sqrt{\frac{1-a^2}{1-a^{2n}}} \sqrt{1-(v/b)^2}, \tag{1.5}$$

where

$$N_{2c} = p_0\sqrt{l(2\pi n)^{-1}}\,\Gamma(1/2+1/2n)\,\Gamma^{-1}(1+1/2n).$$

In the particular case of motion of an isolated crack with $n=1$, we obtain from (1.5)

$$N_2/N_{2c} = \sqrt{1-(v/b)^2};\ N_{2c} = \sqrt{2l}p_0/\pi.$$

The dependences (1.4) and (1.5) of the stress-intensity coefficients for a singularity in the rate of growth of the slit length are shown in Figs. 2 and 3 by curves 1-4, corresponding to the values $n=1, 2, 5, 7$. Superimposed for comparison in these same graphs by dashed curves is the dependence $N/N_c = \sqrt{1-v/b}$ which holds under antiplane deformation for a semiinfinite moving slit in the case of loading by time-independent forces [7]. It is seen in Figs. 2 and 3 that the finiteness of the slits in the problem under consideration results in an increase in the ratio N_i/N_{ic} ($i=1, 2$), where the larger the number of cracks in the system, the higher the degree.

§2. A class of problems in which the displacements are homogeneous functions of the coordinates and time was considered in [1, 4], devoted to self-similar problems of the plane theory of elasticity.

For the problem under consideration this case is realized if the load on the slit edges is representable in the form $\mathbf{p} = p_0 t_0/t \cdot f(\rho/(bt)) \cdot \mathbf{k}$. For the analytic function $U(z_2)$, whose real part is the displacement $w(x_2, y_2)$, we obtain the Keldysh–Sedov boundary-value problem in the z_2 plane:

$$\operatorname{Im} U'(z_2) = 2p_0 b t_0/\mu \cdot f(\xi)(1+r^2)^{-1}|dz_1/dz_2| \text{ for } y_2 = 0,\ 0 < |x_2| < x_{21};$$
$$\operatorname{Re} U'(z_2) = 0 \text{ for } y_2 = 0,\ |x_2| > x_{21}.$$

The solution of this problem which possesses the necessary singularity at the noses of the slits and the correct behavior at infinity can be represented as

$$dU/dz_2 = -\frac{2}{\pi\mu} \frac{p_0 b t_0}{\sqrt{z_2^2 - x_{21}^2}} \int_{-x_{21}}^{+x_{21}} \frac{f(s)}{1+r^2(s)} \frac{\sqrt{s^2 - x_{21}^2}}{(s-z_2)} |dz_1/dz_2|\,ds.$$

Let us examine a particular form of the load:

$$f(x_2) = \begin{cases} 1, & |x_2| < x_{21}^0, \\ 0, & x_{21}^0 < |x_2| < x_{21}, \end{cases}$$

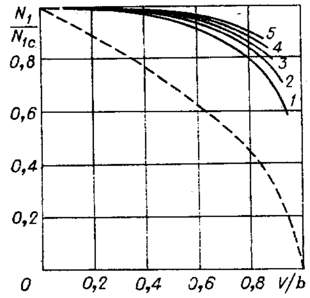

Fig. 4

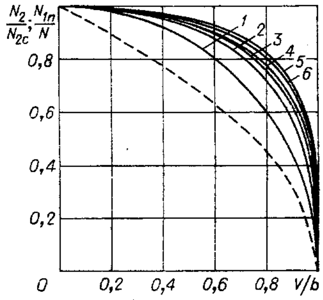

Fig. 5

where $x_{21}^0 = 2r_0^n(1+r_0^{2n})^{-1}$; $r_0 = (b/v_0)\,[1-\sqrt{1-(v_0/b)^2}]$; $v_0 \le v$. For symmetric loading (folding of the angle), the coefficient of the singularity in the stress field is given by the formula

$$N_1/N_{1c} = 2\sqrt{2/\pi}\, I_1 \sqrt{b/v} \cdot \frac{\Gamma(1/2+1/2n)}{\Gamma(1+1/2n)}\, a^{n-1/2} \frac{\sqrt{(1-a^2)(1-a^{2n})}}{(1+a^{2n})^{3/2}}, \tag{2.1}$$

$$I_1 = \frac{1+a^{2n}}{2a^n} \int_0^{r_0} \frac{(1+r^{2n})\,dr}{(1+r^2)\sqrt{(a^{-2n}-r^{2n})(a^{2n}-r^{2n})}}.$$

In the case of loading, for which the displacements w(x, y, t) are uneven relative to the bisecting angle (torsion of the angle), the coefficient of the characteristic field is given by the formula

$$N_2/N_{2c} = 2\sqrt{2/\pi}\, I_2 \sqrt{b/v} \cdot n a^{-1/2} \frac{\Gamma(1+1/2n)}{\Gamma(1/2+1/2n)} \sqrt{\frac{(1-a^2)(1-a^{2n})}{1+a^{2n}}}, \tag{2.2}$$

$$I_2 = \frac{1+a^{2n}}{a^n} \int_0^{r_0} \frac{r^n\,dr}{(1+r^2)\sqrt{(a^{-2n}-r^{2n})(a^{2n}-r^{2n})}},$$

where N_{1c} and N_{2c} in (2.1) and (2.2) are given by (1.4) and (1.5), but it is hence necessary to replace p_0 by p_0t_0/t.

Curves 1-5 in Fig. 4 show the change in the ratio N_1/N_{1c} as a function of v/b for n = 1, 2, 3, 4, 5, respectively. It is assumed that the stresses act along the whole length of the cracks, i.e., $v_0 = v$. The integral I_1 is computed numerically. As is seen from the curves presented, as $n \to \infty$ the magnitude of the ratio $N_1/N_{1c} \to 1$ for $v/b < 1$.

Curves 5 and 6 in Fig. 5 show the change in the ratio N_2/N_{2c} as a function of v/b under the condition that the stresses act on the whole length of the crack. Curve 5 corresponds to n = 1 and curve 6, to n = 5. The integral I_2 in (2.2) is computed numerically.

It is interesting to examine the limit case obtained from (2.1) if v_0 is allowed to tend to zero, but hence $p_0 \to \infty$ in such a way that $2p_0v_0t_0 = Q = \text{const}$. This case corresponds to loading by lumped forces acting at the vertices of wedges cut by cracks. The expression

$$N_{1n} = Q/\pi \cdot \sqrt{n/(2l)}\,[1-(v/b)^2]^{1/4} \sqrt{(1-a^{2n})/(1+a^{2n})} \tag{2.3}$$

is obtained for the stress-intensity coefficients at the vertices of the cracks. This expression is simplified substantially in the limit case n = 1 and $n \gg 1$ and $v/b < 1$:

$$\begin{aligned} N_{11} &= Q/\pi \cdot (2l)^{-1/2} \sqrt{1-(v/b)^2} \quad \text{for } n = 1; \\ N_{1n} &= Q/\pi \cdot \sqrt{n/(2l)}\, \sqrt[4]{1-(v/b)^2} \quad \text{for } n \gg 1. \end{aligned} \tag{2.4}$$

The quantity $N = Q/\pi \cdot \sqrt{n/(2l)}$ is the solution of the static problem of a star crack at whose vertices of the angles the lumped forces Q act. From this and from the equalities (2.3) and (2.4) it follows that the magnitude of the ratio N_{1n}/N varies between $[1-(v/b)^2]^{1/2}$ for n = 1 and $[1-(v/b)^2]^{1/4}$ as $n \to \infty$ (Fig. 5, curves 1 and 4). Curves 2 and 3 in Fig. 5 correspond to the values n = 2 and 3. Computations performed by means of (2.3) showed that even for n = 4 the appropriate curve agrees with the limit $[1-(v/b)^2]^{1/4}$ for practically all values of v/b.

The exact solutions obtained permit a qualitative estimation of the influence of the loading method, the number of cracks, and the velocity of their motion in the development of a radical system of cracks which originates during the explosion of a high-explosive cord charge in a frangible medium.

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